Analysis of the Generalized Inverse Polynomial Scheme to Initial Value Problems

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Abstract—In this paper, we study the analysis of the generalized inverse polynomial scheme for the numerical solution of initial value problems of ordinary differential equation. At first, we generalize the scheme up to the fifth stage using the Binomial expansion and Taylor’s series method towards its derivation. The trend shows the generalization to the \( k^{th} \) term. The analysis demonstrates the efficiency and the effectiveness of the generalized scheme.

Keywords—Taylor’s Series, Initial Value Problem, Stability, Consistency, Convergence

1 INTRODUCTION

In science and engineering, modeling systems theory frequently amounts to solving an initial value problem. In this context, the differential equation is an evolution equation specifying how, given initial conditions, the system will evolve with time. Turning the rules that govern the evolution of a quantity into a differential equation is called modeling (Blanchard, et al., 2012). Numerical method is a substantial aspect in solving initial value problems in ordinary differential equations where the problems cannot be solved or difficult to obtain analytically. The numerical solutions of first order initial value problems have caught much attention recently; a new numerical scheme for the solution of initial value problems in ordinary differential equations is used in (Sunday and Odekonle, 2012) where an integrator was developed by representing the theoretical solution to initial value problems by an interpolating function which may be linear or nonlinear. Also, (Abolarin and Akingbade, 2015) developed a new scheme that takes care of any form of initial value problems. There is a technique for comparing numerical methods that have been designed to solve stiff systems of ordinary differential equations; the technique was applied to five methods of which three turn out to be quite good. However, each of the three has a weakness of its own, which can be identified with particular problem characteristics (Enright, et al., 1975).

Wavelet was used in solving the first order ordinary differential equations which are either stiff or non-stiff (Hsiao, 2005). It is worth mentioning that (Abramov and Yukhno, 2013) work on the numerical solution of the Painlevé equations (equations having singularities at points where the solution takes certain finite values). Researchers also worked on some other forms of equations like the integro-differential equations (Asgari, 2015) and the integral equations where the Petrov-Galerkin method is employed for the numerical solution of stochastic Volterra integral equations (Hosseini, et al., 2015). Other notable works are (Dingwen Deng and Tingting Pan, 2015), (Ogunrinde, 2015), (Wambeqc, 2012), (Okosun, 2003), (Kama and Ibijola, 2000) (Fatunla, 1988) to mention a few.

In (Okosun, 2003), the effectiveness of the first stage, second stage of the inverse polynomial method to solving ordinary differential equations with singularities was shown using a distinct integrator and (Abolarin and Akingbade, 2017) worked on the fourth stage of the method and analyzed its analytical properties; the local truncation error and the order were also determined towards its implementation.

In this paper, our aim is to study the analysis of the generalized inverse polynomial scheme which makes it to be effective and efficient for all stages. To do this, we generalize the numerical integrator in (Abolarin and Akingbade, 2017) up to the fifth term, which is capable of solving equations of the form

\[ y' = f(x, y); y(x_0) = y_0 \] (1)

The integrator is developed by representing the theoretical solution, \( y(x) \) in (1) by an interpolating function.

2 PRELIMINARIES

In this section, we present some useful existing concept and works.

**Definition 1**: The conventional one-step numerical integrator for initial value problem (1) is generally described according to (Lambert, 1973) as

\[ y_{n+1} = y_n + h\phi(x_n, y_n; h) \] (2)

Examples of such method are Euler’s method and Runge-Kutta’s method.

**Definition 2**: Truncation error is the error committed when the higher terms of the power series are ignored. Such errors are essentially algorithmic errors and we can predict the extent of the error that will occur in the method.

**Definition 3**: An algorithm is said to be numerically stable if an error whatever the cause does not grow much larger during calculation. This happens if the problem is well posed, that is, the solution changes by only a small amount if the problem data are changed by small amounts.

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Definition 4: The simplest methods of order ‘p’ are always based on the Taylor series expansion of the solution, $y(x)$ of the IVP (1). If we assume $y^{p+1}(x)$ to be continuous on the closed interval $[a, b]$, then the Taylor’s formula is given by

$$y(x_{n+1}) = y(x_n + h) = y(x_n) + hy'(x_n) + \frac{h^2}{2!}y''(x_n) + \ldots + \frac{h^p}{p!}y^{(p)}(x_n) + O(h^{p+1})$$

where $x_n < \xi_n \leq x_{n+1}$. (3)

The continuity of $y^{p+1}(x)$ implies that it is bounded on $[a, b]$ and therefore,

$$\frac{y^{p+1}(\xi_n)h^{p+1}}{(p + 1)!} = 0(h^{p+1})$$

We introduce (4) in (3),

$$y(x_n) + h[y'(x_n) + \ldots + \frac{h^{p-1}}{p!}y^{(p)}(x_n)] + O(h^{p+1})$$

Hence,

$$y(x_{n+1}) = y(x_n) + h[f(x_n, y_n)] + \frac{h}{2}f'(x_n, y_n) + \ldots + \frac{h^{p-1}}{p!}f^{(p-1)}(x_n, y_n)] + O(h^{p+1})$$

Equation (5) is called the Taylor Series Method of order ‘p’.

3 THE GENERALIZATION OF THE INVERSE POLYNOMIAL SCHEME UP TO THE FIFTH STEP

Let the numerical approximation $y_{n+1}$ evaluated at $x = x_{n+1}$ to exact solution $y(x_{n+1})$ to the first order ordinary differential equation be represented as

$$y_{n+1} = y_n \left[ \sum_{j=0}^{k} a_j x_n^j \right]^{-1}$$

(6)

The parameters $a_j$ are to be determined from the nonlinear equations that will be generated by considering the following steps as in (Abolarin and Akingbade, 2017):

1. For the kth term, we have

$$y_{n+1} = y_n \left[ 1 + \left( a_{x_n} + a_{x_n^2} + a_{x_n^3} + a_{x_n^4} \right) \right]^{-1}$$

(7)

2. Obtain the Binomial Expansion of (7) up to the fifth term

3. Express the left hand side of (7) in terms of its Taylor’s series expansion

$$y_n + h = y_n + \frac{h y'(x_n)}{2!} + \frac{h^2 y''(x_n)}{3!} + \frac{h^3 y'''(x_n)}{4!} + \frac{h^4 y^{(4)}(x_n)}{5!} + y_{n+1}$$

4. We make the expressions above agree term by term for each parameter.

$$h y_n' = -a_1 x_n y_n$$

$$a_1 = -h y_n'(x_n y_n)^{-1}$$

$$a_2 = \left[ 2h^2 (y_n')^2 - h y_n'' y_n' \right] 2x_n y_n^2 (y_n')^{-1}$$

$$a_3 = \left[ -h^3 y_n''' x_n + 6h y_n'' y_n' x_n - 6h y_n''(y_n')^2 \right] 3x_n y_n^3 (y_n')^{-1}$$

$$a_4 = 4a_1 a_3 + a_2^2 - a_4 + a_4 - 3a_4 a_2 x_n y_n^4$$

Using (8) and (9),

$$a_1 = -\frac{h y_n'}{2x_n y_n} + f(x_n, y_n) + \left( \frac{h y_n'}{x_n y_n} \right)$$

$$a_2 = \left[ -h y_n''' x_n + 6h y_n'' y_n' x_n - 6h y_n''(y_n')^2 \right] 3x_n y_n^3 (y_n')^{-1}$$

$$a_4 = 4a_1 a_3 + a_2^2 - a_4 + a_4 - 3a_4 a_2$$

We express each term of $a_i$ above in relation to (8), (9), and (10)

$$2a_1 a_3 = \frac{h^4 (y_n')^4 - 4h y_n'' (y_n')^3 y_n x_n}{3x_n y_n^3} + \frac{h^4 (y_n')^4}{3x_n y_n^3}$$

$$a_2 = \frac{h^4 (y_n')^4 - 4h y_n'' (y_n')^3 y_n x_n}{3x_n y_n^3} + \frac{h^4 (y_n')^4}{3x_n y_n^3}$$

$$a_4 = 4a_1 a_3 + a_2^2 - a_4 + a_4 - 3a_4 a_2 = \frac{h^4 (y_n')^4}{4x_n y_n^4}$$

So,

$$a_1 = \left[ 24h^4 (y_n')^4 - 36h^2 y_n'' (y_n')^3 y_n^2 + 8h^2 (y_n')^2 y_n'' y_n^2 \right] 4x_n y_n^4$$

$$a_2 = \left[ h^4 y_n''' + 6h (y_n')^2 y_n'' \right] 2x_n y_n^3 (y_n')^2$$

(11)
Similarly,
\[
\begin{align*}
a_i &= \left[ -240 t \left( \frac{y_i}{y} \right)^4 - 60 t \left( \frac{y_i}{y} \right)^3 y_i x_i + 10 t \left( \frac{y_i}{y} \right)^2 y_i x_i^2 + \left( \frac{y_i}{y} \right) y_i x_i^3 \right] \left( \frac{y_i}{y} \right)^4 - 5 \left( \frac{y_i}{y} \right)^4 y_i x_i^4 \right] \frac{y_i}{y} \frac{y_i}{y} \frac{y_i}{y} \\
&= 90 t \left( \frac{y_i}{y} \right)^4 + 20 t \left( \frac{y_i}{y} \right)^3 y_i x_i + 6 t \left( \frac{y_i}{y} \right)^2 y_i x_i^2 + h \left( \frac{y_i}{y} \right)^2 y_i x_i^3 + 4 \left( \frac{y_i}{y} \right) y_i x_i^4 \right] \left( \frac{y_i}{y} \right)^4 + 5 \left( \frac{y_i}{y} \right)^4 y_i x_i^4 \\
&= \frac{240 t \left( \frac{y_i}{y} \right)^4 y_i x_i^4}{y_i} \frac{y_i}{y} \frac{y_i}{y} \frac{y_i}{y} \\
\end{align*}
\]
(12)

By using (8), (9), (10), (11) and (12) in (7), we have the fifth stage Inverse Polynomial:
\[
\begin{align*}
y_{n+1} &= \left( \frac{5 t y_i}{y} \right)^4 \frac{y_i}{y} \frac{y_i}{y} \frac{y_i}{y} + 120 \left( \frac{y_i}{y} \right)^2 y_i x_i^2 + 120 \left( \frac{y_i}{y} \right)^4 y_i x_i^4 - 60 \left( \frac{y_i}{y} \right)^3 y_i x_i + 120 \left( \frac{y_i}{y} \right)^2 y_i x_i^2 + 20 t \left( \frac{y_i}{y} \right)^4 y_i x_i^3 - 180 \left( \frac{y_i}{y} \right)^3 y_i x_i^2 + 40 t \left( \frac{y_i}{y} \right)^4 y_i x_i^4 - 30 h \left( \frac{y_i}{y} \right)^4 y_i x_i^4 \\
&= 4 \left( \frac{y_i}{y} \right)^4 y_i x_i^4 \frac{y_i}{y} \frac{y_i}{y} \frac{y_i}{y} + 120 \left( \frac{y_i}{y} \right)^2 y_i x_i^2 + 120 \left( \frac{y_i}{y} \right)^4 y_i x_i^4 - 60 \left( \frac{y_i}{y} \right)^3 y_i x_i + 120 \left( \frac{y_i}{y} \right)^2 y_i x_i^2 + 20 t \left( \frac{y_i}{y} \right)^4 y_i x_i^3 - 180 \left( \frac{y_i}{y} \right)^3 y_i x_i^2 + 40 t \left( \frac{y_i}{y} \right)^4 y_i x_i^4 - 30 h \left( \frac{y_i}{y} \right)^4 y_i x_i^4 \\
&= \frac{240 t \left( \frac{y_i}{y} \right)^4 y_i x_i^4}{y_i} \frac{y_i}{y} \frac{y_i}{y} \frac{y_i}{y} \\
\end{align*}
\]
(13)

4 ANALYSIS OF THE GENERALIZED INVERSE POLYNOMIAL SCHEME

4.1 CONSISTENCY

A numerical scheme with an increment function \( \phi(x_n, y_n; h) \) is said to be consistent with the initial value problem (1) if
\[
\phi(x_n, y_n; h) = f(x, y)
\]
when \( h = 0 \).

From (2)
\[
y_{n+1} - y_n = \frac{h}{h} \phi(x_n, y_n; h)
\]
(14)

In light of this, with respect to the scheme,
\[
y_{n+1} - y_n = \left[ 120 t \left( \frac{y_i}{y} \right)^2 y_i x_i^2 + 120 t \left( \frac{y_i}{y} \right)^4 y_i x_i^4 - 60 t \left( \frac{y_i}{y} \right)^3 y_i x_i + 120 t \left( \frac{y_i}{y} \right)^2 y_i x_i^2 + 20 t \left( \frac{y_i}{y} \right)^4 y_i x_i^3 - 180 \left( \frac{y_i}{y} \right)^3 y_i x_i^2 + 40 t \left( \frac{y_i}{y} \right)^4 y_i x_i^4 - \right. \]
\[
\left. 30 h \left( \frac{y_i}{y} \right)^4 y_i x_i^4 - 120 \left( \frac{y_i}{y} \right)^2 y_i x_i^2 + 120 \left( \frac{y_i}{y} \right)^4 y_i x_i^4 - 60 \left( \frac{y_i}{y} \right)^3 y_i x_i + 120 t \left( \frac{y_i}{y} \right)^2 y_i x_i^2 + 20 t \left( \frac{y_i}{y} \right)^4 y_i x_i^3 - 180 \left( \frac{y_i}{y} \right)^3 y_i x_i^2 + 40 t \left( \frac{y_i}{y} \right)^4 y_i x_i^4 - \right. \]
\[
\left. 30 h \left( \frac{y_i}{y} \right)^4 y_i x_i^4 \right] \frac{y_i}{y} \frac{y_i}{y} \frac{y_i}{y} + 4 \left( \frac{y_i}{y} \right)^4 y_i x_i^4 \frac{y_i}{y} \frac{y_i}{y} \frac{y_i}{y} \right]
\]
(15)

4.2 STABILITY

One-step scheme is said to be stable if for any initial error \( e_0 \) there exist a constant \( M \) and \( h > 0 \) such that when the general one-step scheme is applied to initial value problems with step size \( h \in (0, h_0) \) the ultimate error \( e_n \) satisfies the following inequalities
\[
e_n \leq M e_0 \quad \text{and} \quad 0 < M < 1
\]

Using the general form,
\[
y_{n+h} = \frac{1}{h} \sum_{j=0}^{\infty} a_j \left( \frac{y_i}{y} \right)^j \right] \left( \frac{y_i}{y} \right)^4 + 5 \left( \frac{y_i}{y} \right)^4 y_i x_i^4
\]
(16)

The theoretical solution \( y(x) \) is given as
\[
y(x_{n, h}) = \frac{1}{h} \left( \sum_{j=0}^{\infty} a_j y_{n, h}^j \right) + T_{n, h}
\]
(17)

\( y(x_{n, h}) = y(x_{n, h}) \left( \sum_{j=0}^{\infty} a_j x_{n, h}^j \right) + T_{n, h}
\]
\[ e_{n+1} = e_{n+2} \left( \sum_{j=0}^{k} a_j x_{n+1}^j \right)^{-1} + T_{n+1} \]

We take the absolute value of both sides,

\[ |e_{n+1}| = \left| e_{n+2} \left( \sum_{j=0}^{k} a_j x_{n+1}^j \right)^{-1} + T_{n+1} \right| \]

We assume that

\[ Q = \sum_{j=0}^{k} a_j x^j_{n+1} \]

Then,

\[ \left( \sum_{j=0}^{k} a_j x_{n+1}^j \right)^{-1} = \left( \sum_{j=0}^{k} a_j x_{n+1}^j \right)^{-1} \]

\[ \left( \sum_{j=0}^{k} a_j x_{n+1}^j \right)^{-1} = Q^{-1} = M \]

\[ |e_{n+1}| \leq ME_{n+1} + |T_{n+1}| \]

Let \( E_{n+1} = sup(e_{n+1}) \) and \( T = sup(T_{n+1}) \)

Similarly, \( E_{n+2} = sup(e_{n+2}) \) with \( 0 < n < \infty \)

We then have in the form

\[ E_{n+3} \leq ME_{n+2} + T \]

For \( h=1 \),

\[ E_{n+1} \leq ME_{n+1} + T \]

For \( h=2 \),

\[ E_{n+2} \leq ME_{n+1} + T \]

\[ \leq M (ME_{n} + T) + T \]

\[ = M^2 E_n + MT + T \]

\[ E_{n+3} \leq M^3 E_n + MT + T \]

For \( h=3 \),

\[ E_{n+3} \leq ME_{n+2} + T \]

\[ \leq M (M^2 E_n + MT + T) + T \]

\[ E_{n+4} \leq M^4 E_n + M^3 T + MT + T \]

The general form is

\[ E_{n+i} \leq M^i E_n + \sum_{j=0}^{i} M^j T \]

\[ |e_{n+i}| < E_{n+i} \leq M^i E_n + \sum_{j=0}^{i} M^j T \]

Since \( M<1 \) and as \( n \to \infty \), \( E_{n+i} \to 0 \)

This shows that the method is stable and also convergent.

### 4.3 Convergence

The necessary and sufficient conditions for a numerical method to be convergent are stability and consistency. Since these conditions are satisfied, we can conclude that the proposed method is convergent.

### 5 Conclusion

The analysis studied, shows the efficiency of the inverse polynomial scheme and why at any stage implementation, will compete favourably with existing methods as seen in (Abolarin and Akingbade, 2017).

We also generalized up to the fifth stage which definitely has an advantage over all the previously proposed inverse polynomial schemes because of its high order. Having studied the analysis of the generalized scheme, comparative study of the stages will be considered in subsequent work.

### References


